OPTIMAL DESIGN OF PLASTIC SLABS WITH MASS FORCES TAKEN INTO ACCOUNT

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A sufficient criterion for an absolutely minimal weight design in application to plates (disks) of variable thickness H working under plane stress state conditions is established in [1, 2]. By writing the fundamental equations in a coordinate system whose coordinate lines are trajectories of the main stresses, isostats, the authors of [3] showed that four kinds of solutions exist for designs satisfying the condition of constant specific dissipation rate under the Treska fluidity condition. It is proved in [4] that this condition is also a necessary condition for an absolutely minimal weight design for the sides of the Treska hexagon. The characteristics of the equations describing the optimal designs of disks for an arbitrary smooth fluidity condition were studied in [5]. By using available stress fields, the optimal thicknesses of plane elements in the shape of T-plates and extensible polygonal plates with circular and square holes were calculated in [6]. A finite element approach to the problem under consideration is developed in [7]. The mass forces were assumed zero in [3-7].

It should be noted that at this time there are no papers in the literature concerned with taking account of mass forces in the problem of optimal disk design, with the exception of [2] where the particular problem of the minimum weight design for a rotating circular disk is considered for one of the kinds of boundary conditions.

The present paper has the goal of filling this gap somewhat.

1. We take the coordinate plane $x_3^* = 0$ as middle plane. Forces F_1^*, F_2^* independent of H act over part of the boundary Γ_F in the middle plane of the disk. The velocities are zero on the other part of the boundary Γ_U. The mass forces g_1^*, g_2^* referred to unit volume also act in the middle plane. The assumption of a plane stress state implies $\sigma_{13}^* = \sigma_{23}^* = \sigma_{33}^* = 0$. We let u_k^*, σ_{kl}^*, ε_{kl}^*, σ_k^*, ε_k^* denote, respectively, the velocity, stress tensor, strain rate tensor, principal stress, and principal strain rate components. The subscripts k, l later take the values 1, 2 everywhere. We go over to the dimensionless quantities $x_k = x_k^* x_0^{-1}$, $u_k = u_k^* t_0 x_0^{-1}$, $h = HH_0^{-1}$, $\sigma_{kl} = \sigma_{kl}^* \sigma_0^{-1}$, $\sigma_k = \sigma_k^* \sigma_0^{-1}$, $\varepsilon_{kl} = \varepsilon_{kl}^* t_0$, $\varepsilon_k = \varepsilon_k^* t_0$, $F_k = F_k^* \sigma_0^{-1} H_0^{-1}$, $g_k = g_k^* x_0 \sigma_0^{-1}$, where σ_0, t_0, x_0, H_0 are the characteristic stress, time, length, and thickness of the plate. The components are functions of just x_1, x_2 and satisfy the equilibrium equations

$$(\sigma_{kl}h)_{,l} + g_k h = 0 \qquad (1.1)$$

and the boundary conditions on Γ_F

$$(\sigma_{kl}h)n_l = F_k, \qquad (1.2)$$

where n_l are components of the unit normal to the line Γ_F. The plate material is assumed iso-

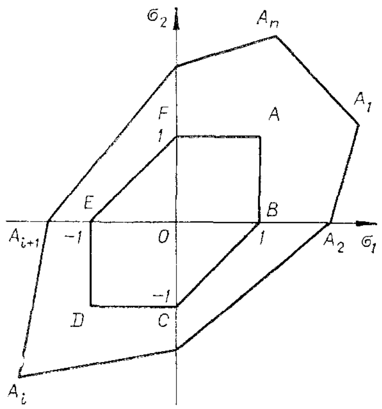

Fig. 1

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tropic, ideally plastic, and with a piecewise-linear potential (Fig. 1). In this connection, two important kinds of optimal designs occur: disks corresponding to the sides A_iA_{i+1} and disks corresponding to the vertex A_i.

The equation of the side A_iA_{i+1} has the form

$$f = a_i\sigma_1 + b_i\sigma_2 = 1, \tag{1.3}$$

where

$$a_i = -Q_i d_i^{-1};\quad b_i = P_i d_i^{-1};\quad P_i = p_{i+1} - p_i; \tag{1.4}$$

$Q_i = q_{i+1} - q_i$, $d_i = p_{i+1}q_i - p_i q_{i+1}$, (p_i, q_i) are the coordinates of the vertex A_i. We write the known relationships [8] as

$$2\{\sigma_{11}, \sigma_{22}\} = \sigma_1 + \sigma_2 \pm (\sigma_1 - \sigma_2)\cos 2\theta,\ 2\sigma_{12} = (\sigma_1 - \sigma_2)\sin 2\theta; \tag{1.5}$$

$$2\{\varepsilon_{11}, \varepsilon_{22}\} = \varepsilon_1 + \varepsilon_2 \pm (\varepsilon_1 - \varepsilon_2)\cos 2\theta,\ 2\varepsilon_{12} = (\varepsilon_1 - \varepsilon_2)\sin 2\theta, \tag{1.6}$$

where θ is the angle between the first principal direction and the x_1 axis. Let us also write down the Cauchy formula

$$2\varepsilon_{kl} = u_{k,l} + u_{l,k}. \tag{1.7}$$

2. We consider the optimal designs corresponding to the side A_iA_{i+1}. The flow law for the side A_iA_{i+1} has the form

$$\varepsilon_1 = \lambda a_i,\ \varepsilon_2 = \lambda b_i,\ \lambda \geqslant 0. \tag{2.1}$$

The condition for constant modified dissipative function is $\Delta = \sigma_k\varepsilon_k - g_k u_k = \text{const}$, from which

$$\lambda = g_k u_k + \Delta,\ \Delta = \text{const}. \tag{2.2}$$

Equations (1.1), (1.3), (1.5)-(1.7), (2.1), (2.2) form a closed system of fifteen equations with fifteen unknown functions. We show that in the case of constant mass forces this system is successfully reduced to a system of four quasilinear first-order partial differential equations.

Let g_k be certain constants. We introduce the notation

$$\omega = 0.5\lambda^{-1}(u_{1,2} - u_{2,1}),\ s_i = 0.5(b_i - a_i),\ t_i = 0.5(b_i + a_i). \tag{2.3}$$

By virtue of (1.6), (1.7), (1.9), (2.1)-(2.3), we have

$$\{\varepsilon_{11}, \varepsilon_{22}\} = \lambda(t_i \mp s_i\cos 2\theta),\ \varepsilon_{12} = -\lambda s_i \sin 2\theta; \tag{2.4}$$

$$\lambda_{,1} = g_1\varepsilon_{11} + g_2(\varepsilon_{12} - \omega\lambda),\ \lambda_{,2} = g_1(\varepsilon_{12} + \omega\lambda) + g_2\varepsilon_{22}. \tag{2.5}$$

Differentiating ω with respect to x_1, x_2, replacing the partial derivatives of u_k by using (1.7), and then utilizing (2.4) and (2.5), we arrive at the system

$$\begin{gathered} 2s_i(\theta_{,1}\cos 2\theta + \theta_{,2}\sin 2\theta) - \omega_{,1} = -g_2(\omega^2 + t_i^2 - s_i^2), \\ 2s_i(\theta_{,2}\cos 2\theta - \theta_{,1}\sin 2\theta) + \omega_{,2} = -g_1(\omega^2 + t_i^2 - s_i^2). \end{gathered} \tag{2.6}$$

If the solution ω, θ of the system (2.6) is found, then we obtain a system to find the velocity from (1.7), (2.3), (2.4)

$$u_{k,l} = U_{kl}\lambda, \text{ where}$$
$$U_{kk} = t_i + (-1)^k s_i\cos 2\theta;\ U_{kl} = (-1)^l\omega - s_i\sin 2\theta;\ k \neq l. \tag{2.7}$$

From (2.2) and (2.7) we have

$$\lambda_{,k} = V_k\lambda, \text{ where } V_k = g_l U_{lk}. \tag{2.8}$$

It is easy to see that the system (2.8) and the system (2.7) are compatible if only ω, θ is a solution of (2.6). We write the system (2.8) in the form $(\ln\lambda)_k = V_k$, after which we find $\ln\lambda$, and then u_k from (2.7) by means of their total differentials. Therefore, taking account of (2.4) and (2.5) the system (2.6) is the strain compatibility condition. In the case $s_i \neq 0$ it contains two unknown functions and in the case $s_i = 0$ (the segment A_iA_{i+1} is parallel to the line $\sigma_1 + \sigma_2 = 0$) just one unknown function ω. We consequently consider these two cases separately.

In the case $s_i \neq 0$ we set $\chi = 0.5(\sigma_1 + \sigma_2)$, then

$$\{\sigma_{11}, \sigma_{22}\} = \chi \pm (C_i\chi + D_i) \cos 2\theta, \ \sigma_{12} = (C_i\chi + D_i) \sin 2\theta, \tag{2.9}$$

where $C_i = t_i s_i^{-1}$, $D_i = -0.5 s_i^{-1}$. Substituting (2.9) into (1.1) and taking account of (2.6), we arrive at the system

$$\begin{gathered} h_{,1}[\chi + (C_i\chi + D_i)\cos 2\theta] + h_{,2}(C_i\chi + D_i)\sin 2\theta + \chi_{,1}(1 + C_i\cos 2\theta) + \\ + \chi_{,2}C_i\sin 2\theta - s_i^{-1}(C_i\chi + D_i)\left[\omega_{,2} + g_1(\omega^2 + t_i^2 - s_i^2)\right]h + g_1 h = 0, \\ h_{,1}(C_i\chi + D_i)\sin 2\theta + h_{,2}[\chi - (C_i\chi + D_i)\cos 2\theta] + \chi_{,2}(1 - C_i\cos 2\theta) + \chi_{,1}C_i\sin 2\theta + \\ + s_i^{-1}(C_i\chi + D_i)\left[\omega_{,1} - g_2(\omega^2 + t_i^2 - s_i^2)\right]h + g_2 h = 0. \end{gathered} \tag{2.10}$$

Therefore, the problem is reduced to the solution of two systems of quasilinear equations (2.6) and (2.10). A system of inequalities

$$h \geqslant 0, \ P_i p_i + Q_i q_i \leqslant P_i\sigma_1 + Q_i\sigma_2 \leqslant P_i p_{i+1} + Q_i q_{i+1}, \ \Delta > 0. \tag{2.11}$$

should also be appended to the system of equations obtained.

The differential operator in the left side of (2.6) agrees, to the accuracy of the notation, with the corresponding operator of the system of plain strain equations of an ideally plastic body [8]. Therefore, the system (2.6) is hyperbolic, its characteristic directions are $\gamma_1 = \mathrm{tg}\,\theta$, $\gamma_2 = -\mathrm{ctg}\,\theta$. The equation to find the characteristics of the system (2.10) ($\gamma = dx_2/dx_1$) has the form

$$\begin{vmatrix} -\gamma[\chi + (C_i\chi + D_i)\cos 2\theta] + & -\gamma(1 + C_i\cos 2\theta) + \\ + (C_i\chi + D_i)\sin 2\theta, & + C_i\sin 2\theta, \\ -\gamma(C_i\chi + D_i)\sin 2\theta + \chi - & -\gamma C_i\sin 2\theta - 1 - \\ - (C_i\chi + D_i)\cos 2\theta, & - C_i\cos 2\theta, \end{vmatrix} = 0,$$

from which we obtain $\gamma_3 = \mathrm{tg}\,\theta$, $\gamma_4 = -\mathrm{ctg}\,\theta$. Therefore, the system (2.16), (2.10) is hyperbolic with four real families of characteristics agreeing with the isostats.

Let f_1, f_2 be corresponding right sides in (2.8). The relationships on the characteristics (2.8) have the form [9]

$$dx_2 = \mathrm{tg}\,\theta dx_1, \ d(\omega - 2s_i\theta) + f_1 dx_1 - f_2 dx_2 = 0,$$

$$dx_2 = -\mathrm{ctg}\,\theta dx_1, \ d(\omega + 2s_i\theta) + f_1 dx_1 - f_2 dx_2 = 0.$$

In the general case the system (2.6), (2.10) can be solved by using numerical methods [8, 10]. However, in certian cases an analytic solution is obtained successfully. Let us note that for $g_1 = g_2 = 0$ the relationships on the characteristics (2.6) are represented in the form of total differentials. By analogy with this, we consider the situation when the expression $f_1dx_1 - f_2dx_2$ is a total differential for nonzero g_k. From this condition we obtain the equation

$$\omega(g_1\omega_{,1} + g_2\omega_{,2}) = 0,$$

whose general solution is

$$\omega = \omega^0(g_2x_1 - g_1x_2),$$

where $\omega^0(t)$ is a certain function. Here $f_1dx_1 - f_2dx_2 = dG$, $G = G^0(g_2x_1 - g_1x_2)$ and $G^0(t)$ is the solution of the equation

$$G^{0\prime}(t) = -\omega^{02}(t) - s_i^2 - t_i^2.$$

Setting $p^0(t) = \omega^0(t) + G^0(t)$ and $p = p^0(g_2x_1 - g_1x_2)$, we reduce the system (2.6) to the form

$$2s_i(\theta_{,2}\sin 2\theta + \theta_{,1}\cos 2\theta) - p_{,1} = 0, \ 2s_i(\theta_{,2}\cos 2\theta - \theta_{,1}\sin 2\theta) + p_{,2} = 0. \tag{2.12}$$

If p, θ is known to be a solution of (2.12), then $\omega^0(t)$ is found from the solution of the ordinary differential equation

$$\omega^{0\prime}(t) - \omega^{02}(t) = p^{0\prime}(t) + t_i^2 - s_i^2,$$

whose general solution [11] is omitted because of its awkwardness.

The equations for the characteristics of the system (2.12) are

$$dx_2 = \operatorname{tg}\theta dx_1,\ \xi = p - 2s_i\theta,\ dx_2 = -\operatorname{ctg}\theta dx_1,\ \eta = p + s_i\theta.$$

Selecting ξ, η as new unknown functions, we convert (2.12) into the equivalent system

$$\xi_{,1} + \xi_{,2}\operatorname{tg}\theta = 0,\ \eta_{,1}\operatorname{tg}\theta - \eta_{,2} = 0. \tag{2.13}$$

In the case when the Jacobian

$$J = 2\xi_{,1}\eta_{,1}(\sin 2\theta)^{-1} = -2\xi_{,2}\eta_{,2}(\sin 2\theta)^{-1}$$

is not zero, replaced of the unknown functions by the independent variables, the system (2.13) can be reduced by a linear system [8]. This latter can be solved by numerical methods either by using trigonometric series or approximate integration. Some simple solutions are obtained when J = 0: 1) ξ, η — const; 2) ξ — const; 3) η — const. Let us examine just the first case $\xi = \xi_0$, $\eta = \eta_0$. Then evidently $\theta = \theta_0$, $p = p_0$, and $\omega = \omega^\circ(g_2x_1 - g_1x_2)$, where the function $\omega^\circ(t)$ is a solution of the equation

$$\omega^{0\prime}(t) - \omega^{02}(t) = t_i^2 - s_i^2,$$

whose general solution is

$$\omega^0(t) = \begin{cases} \sqrt{a_ib_i}\operatorname{tg}\left(\sqrt{a_ib_i}\,t + c\right), & \text{if } a_ib_i > 0, \\ \sqrt{-a_ib_i}\,\dfrac{c\exp\left(-2\sqrt{-a_ib_i}\,t\right)+1}{c\exp\left(-2\sqrt{-a_ib_i}\,t\right)-1}, & \text{if } a_ib_i < 0, \\ 0 \text{ or } (c-t)^{-1}, & \text{if } a_ib_i = 0 \end{cases} \tag{2.14}$$

(c is the constant of integration).

Let us introduce new independent variables $y_k = n_{kl}x_l$, where $n_{11} = n_{22} = \cos\theta_0$, $n_{12} = -n_{21} = \sin\theta_0$. It is easy to reduce the system (2.10) to the form

$$\frac{\partial}{\partial y_1}\{[(1 + C_i)\chi + D_i]\} + g_1^0h = 0,\quad \frac{\partial}{\partial y_2}\{[(1 - C_i)\chi - D_i]h\} + g_2^0h = 0,$$

where $g_k^\circ = n_{kl}g_l$. This last system is none other than the system of equilibrium equations in the coordinates y_1y_2

$$\frac{\partial}{\partial y_k}(\sigma_kh) + g_k^0h = 0. \tag{2.15}$$

Later, for simplicity $g_1^\circ = 0$. From the first equation in (2.15), $T_1 = \sigma_1h = Y_2(y_2)$, where $Y_2(y_2)$ is a certain function.

If $b_i \neq 0$, then $T_2 = \sigma_2h = b_i^{-1}h - a_ib_i^{-1}Y_2(y_2)$. We substitute this expression into the second equation in (2.15). After integrating we obtain

$$h = \left(Y_1(y_1) + a_i\int_{y_2^0}^{y_2} Y_2'(y_2)\exp\left(b_ig_2^0y_2\right)dy_2\right)\exp\left(-b_ig_2^0y_2\right),$$

where $Y_1(y_1)$ is a certain function.

If $b_i = 0$, then $\sigma_1 \equiv 1/a_i$. From (2.15) we obtain

$$T_1 = Y_2(y_2),\ T_2 = a_ig_2^0\left[Y_1(y_1) - \int_{y_2^0}^{y_2} Y_2(y_2)\,dy_2\right],\quad h = a_iY_2(y_2), \tag{2.16}$$

where $T_k = \sigma_kh$ and $Y_k(y_k)$ are certain functions.

The functions $Y_k(y_k)$ are determined from the boundary conditions and should be subject to the system (2.11). For example, for $b_i = 0$ we have

$$a_iY_2(y_2) \geqslant 0,\quad q_iQ_i \leqslant g_2^0Q_i\left[Y_1(y_1) - \int_{y_2^0}^{y_2} Y_2(y_2)\,dy_2\right]Y_2^{-1}(y_2) \leqslant q_{i+1}Q_i,\quad \Delta > 0. \tag{2.17}$$

For $s_i = 0$ we set $\chi = 0.5(\sigma_1 - \sigma_2)$, then

$$\{\sigma_{11}, \sigma_{22}\} = m_i \pm \chi\cos 2\theta,\quad \sigma_{12} = \chi\sin 2\theta,\quad m_i = 0.5b_i^{-1}. \tag{2.18}$$

We obtain two systems of equations from (1.1), (2.18), and (2.6):

$$\omega_{,1} = g_2(\omega^2 + t_i^2), \quad \omega_{,2} = -g_1(\omega^2 + t_i^2); \tag{2.19}$$

$$\begin{gathered} h_{,1}(m_i + \chi\cos 2\theta) + h_{,2}\chi\sin 2\theta + \chi_{,1}h\cos 2\theta + \chi_{,2}h\sin 2\theta + 2\chi h(\theta_{,2}\cos 2\theta - \\ - \theta_{,1}\sin 2\theta) + g_1 h = 0, \\ h_{,1}\chi\sin 2\theta + h_{,2}(m_i - \chi\cos 2\theta) + \chi_{,1}h\sin 2\theta - \chi_{,2}h\cos 2\theta + 2\chi h(\theta_{,1}\cos 2\theta + \\ + \theta_{,2}\sin 2\theta) + g_2 h = 0. \end{gathered} \tag{2.20}$$

It is easy to find the general solution of the system (2.19)

$$\omega = t_i \operatorname{tg}[(g_2 x_1 - g_1 x_2)t_i + c],$$

where c is the constant of integration. Furthermore, the system (2.20) contains two equations and three unknown functions. Consequently, one of the functions, say θ, can be given arbitrarily and the linear system of differential equations can be solved in terms of the two other functions. If θ is given, then the boundary conditions on Γ_F for the system (2.20) will be determined from the system (1.2) and (2.18). Let us note that there are no constraints on the function θ. Therefore, in the case $s_i = 0$ ambiguity of the solution of the problem formulated above should be observed as was noted in [12].

The equation for the characteristics of the system (2.20) (θ is a known function) has the form $m_i h \sin 2\theta(\gamma^2 + 2\gamma \operatorname{ctg} 2 \quad - 1) = 0$, from which $\gamma_1 = \operatorname{tg}\theta$, $\gamma_2 = -\operatorname{ctg}\theta$. Therefore, this system is hyperbolic; its characteristics agree with the isostats. In conclusion, we note that the system of inequalities (2.11) should be adjoined to the system (2.19) and (2.20) as for $s_i \neq 0$.

3. Let us examine optimal designs corresponding to the vertex A_i. In this case $\sigma_1 = p_i$, $\sigma_2 = q_i$. From (1.1) and (1.5) we obtain

$$\begin{gathered} h_{,1}[(p_i + q_i) + (p_i - q_i)\cos 2\theta] + h_{,2}(p_i - q_i)\sin 2\theta + 2h(p_i - q_i)(\theta_{,2}\cos 2\theta - \theta_{,1}\sin 2\theta) + 2g_1 h = 0, \\ h_{,1}(p_i - q_i)\sin 2\theta + h_{,2}[(p_i + q_i) - (p_i - q_i)\cos 2\theta] + 2h(p_i - q_i)(\theta_{,1}\cos 2\theta + \theta_{,2}\sin 2\theta) + 2g_2 h = 0. \end{gathered} \tag{3.1}$$

Therefore, the system (3.1) is closed with respect to h, θ. For the vertex B of the Treska hexagon (see Fig. 1), it is considered in 4 in the absence of mass forces. It is true that the coefficients are written down incorrectly (for $h_{,1}$ in the first equation and for $h_{,2}$ in the second the constant components $p_i + q_i$ were lost), which affected the type and subsequent solution of the system.

The flow law for the vertex A_i has the form

$$\varepsilon_1 = \lambda[\mu a_{i-1} + (1-\mu)a_i], \ \varepsilon_2 = \lambda[\mu b_{i-1} + (1-\mu)b_i], \ 0 \leqslant \mu \leqslant 1, \ \lambda > 0.$$

From the optimality condition $\lambda = g_k u_k + \Delta$, $\Delta = \text{const}$. The mass forces are not generally assumed constant. Investigation of the system is substantially different in the following two cases.

1. If $p_i \neq q_i$, the equation of the characteristics (3.1) will be

$$2h(p_i - q_i)[(p_i + q_i)\cos 2\theta + (p_i - q_i)]\gamma^2 - 2(p_i + q_i)\gamma\sin 2\theta + (p_i - q_i) - (p_i + q_i)\cos 2\theta = 0,$$

from which

$$\gamma_{1,2} = [(p_i + q_i)\sin 2\theta \pm 2\sqrt{p_i q_i}]/[(p_i + q_i)\cos 2\theta + (p_i - q_i)].$$

Therefore, the system (3.1) is hyperbolic for $p_i q_i > 0$, parabolic for $p_i q_i = 0$, and elliptical for $p_i q_i < 0$.

2. If $p_i = q_i$, we have the system

$$p_i h_{,1} + g_1 h = 0, \ p_i h_{,2} + g_2 h = 0 \tag{3.2}$$

for one unknown function. We write (3.2) in the form

$$(\ln h)_{,1} + g_1 p_i^{-1} = 0, \quad (\ln h)_{,2} + g_2 p_i^{-1} = 0.$$

For compatiblity of (3.2), it is therefore necessary that $g_{1,2} = g_{2,1}$. Upon satisfying this last condition we find ln h by its total differential. When $g_1 = g_2 = 0$ a disk of constant thickness will be the solution as in [3, 4].

4. As an illustration we examine a variable thickness rectangular slab one of whose sides is clamped in a vertical wall (Fig. 2). A normal force of intensity $Y(x_2)$ acting in

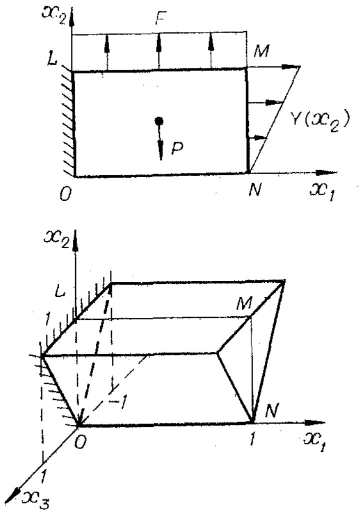

Fig. 2

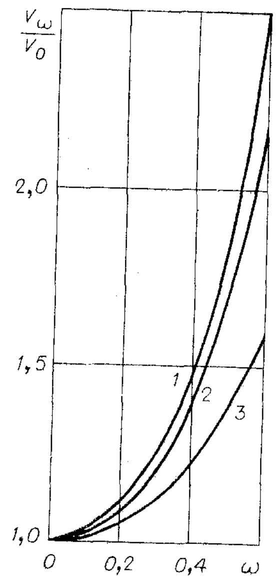

Fig. 3

the plane of the slab is applied to the opposite side. The side ON is force-free and uniformly distributed normal force of intensity F is applied to the side LM. A mass force **P**: $g_1 = 0$, $g_2 = -\rho$, where ρ is the density of the slab material, also acts on the slab.

We assume that the regime AB of the Treska plasticity condition is realized in an optimal plate (see Fig. 1). Then $a_i = 1$, $b_i = 0$. We set $\theta = 0$ and $\omega = 0$ by virtue of (2.14). Hence $g_i^0 = 0$ and $p = p_0$, consequently, $\xi = \xi_0$, $\eta = \eta_0$. Therefore, the optimal design is determined by (2.16). From the boundary conditions we obtain the forces T_k and the relation between F and $Y(x_2)$

$$T_1 = h = Y(x_2), \quad T_2 = \rho \int_0^{x_2} Y(x_2)\, dx_2, \quad F = \rho \int_0^{b} Y(x_2)\, dx_2, \quad b = |OL|.$$

The inequalities (2.17) yield constraints on $Y(x_2)$

$$Y(x_2) \geqslant 0, \quad \rho \int_0^{x_2} Y(x_2)\, dx_2 \leqslant Y(x_2), \quad 0 \leqslant x_2 \leqslant b.$$

Furthermore, $U_{11} = 1$, $U_{22} = U_{12} = U_{21} = V_1 = V_2 = 0$, hence $\lambda = \lambda_0 = \text{const}$. Taking account of the boundary conditions, we obtain $u_1 = \lambda_0 x_1$, $u_2 = 0$ from (2.7).

For a specific example, we set $|OL| = |ON| = 1$, $\rho = 1$, $F = 1$, $\lambda_0 = 1$, $Y(x_2) = 2x_2$. Here $h = 2x_2$ (see Fig. 2).

5. A particular case of the problem under consideration is the problem of finding the minimal volume of a rotating disk.

A circular annular disk with radius R_1 of the inner circle and R_2 of the outer circle is rotated at a constant angular velocity ω^* around an axis perpendicular to the plane of the disk and passing through its center. The disk boundaries are loaded by uniformly distributed forces of intensity T_1^* and T_2^*, or forces are given on one boundary while the velocities equal zero on the other. Find the disk thickness corresponding to minimal volume.

Let r^*, u_r, σ_r, σ_θ, ε_r, ε_θ, ρ^*, respectively, be the radius, radial velocity, principal stresses, principal strain rates, and material density of the disk. Let us turn to dimensionless: $r = r^* r_0^{-1}$, $u = u_r t_0 r_0^{-1}$, $\omega = \omega^* t_0$, $\sigma_1 = \sigma_r \sigma_0^{-1}$, $\sigma_2 = \sigma_\theta \sigma_0^{-1}$, $\varepsilon_1 = \varepsilon_r t_0$, $\varepsilon_2 = \varepsilon_\theta t_0$, $\rho = \rho^* r_0^2 t_0^{-2} \sigma_0^{-1}$, $h = H H_0^{-1}$, $r_k = R_k r_0^{-1}$, $T_k = T_k^* \sigma_0^{-1} H_0^{-1}$, where σ_0, t_0, r_0, H_0 are the characteristic stress, time length, and thickness of the disk. The stresses σ_k satisfy the equilibrium equation

$$(h\sigma_1)_{,r} + h(\sigma_1 - \sigma_2)/r = -\rho\omega^2 r h, \tag{5.1}$$

where the comma denotes differentiation with respect to r. The strain rate components are expressed in terms of u

$$\varepsilon_1 = u_{,r}, \quad \varepsilon_2 = u/r; \tag{5.2}$$

and we have for the quantity Δ

$$\Delta = \sigma_k \varepsilon_k - \rho\omega^2 ru.$$

The case $T_1 = 0$, $T_2 > 0$ was considered in [2] under Treska fluidity conditions. The optimality condition imposes a constraint on the set of allowable locations of the stress points on the flow hexagon. It is clarified that only the stresses representable by the points A, B, D, E can correspond to the velocity field subject to the condition Δ = const. The authors of [2] excluded the points D and E from consideration since $T_1 = 0$, $T_2 > 0$ and the point B since $u = r\varepsilon_2 \leqslant 0$ on the inner boundary in this case. It is impossible to agree with the last remark since the condition $u \leqslant 0$ is nowhere contradicted. An example will be presented below that shows that optimal designs exist for other boundary conditions, that work in the regime B for which this condition is nevertheless satisfied. In the case $T_1 = 0$ the point B should indeed be excluded from the considerations but for another reason: The general solution of (5.1) for the regime B is

$$h = h_0 r^{-1} \exp(-0.5\rho\omega^2 r^2), \quad h_0 = \text{const}, \tag{5.3}$$

consequently, $h \equiv 0$ in the plate follows from the condition $T_1 = 0$. For this same reason, regime A is indeed impossible for $T_1 = 0$. Therefore, optimal solutions are not constructed successfully for the boundary conditions mentioned. It must be said that the very same condition $T_1 = 0$ is rather exaggerated for the problem of a rotating disk and does not reflect the situation that is observed in real structures of this kind: either the displacement is zero on the inner boundary or a force different from zero is given. The authors tried to emerge artificially from the contradiction obtained by introducing and attaching an unreal flange of infinite height to the disk, but of finite meridian section area. Moreover, by imposing the additional condition $u_{,r} = 0$ for $r = r_1$, which does not result from the formation of the problem, the authors of [2] lost an arbitrary constant for integrating the equations for the velocities. This resulted in a constraint on the angular velocity of the disk $\rho\omega^2 r_1^2 < 1$.

Let us construct examples of optimal disks for more natural boundary conditions for the same flow conditions.

Let us consider the following boundary conditions

$$u(r_1) = 0, \quad (\sigma_1 h)(r_2) = T > 0. \tag{5.4}$$

Assuming the optimal design of the disk to operate in the regime A, we have from (5.1) and the boundary condition for $r = r_2$

$$h = T \exp\left[0.5\rho\omega^2 \left(r_2^2 - r^2\right)\right]. \tag{5.5}$$

Then from the optimality condition

$$u_{,r} + u/r - \rho\omega^2 ru = \Delta, \quad \Delta = \text{const} > 0$$

and the boundary condition for $r = r_1$ we obtain

$$u = \begin{cases} \Delta\rho^{-1}\omega^{-2}r^{-1}\left\{\exp\left[0.5\rho\omega^2\left(r - r_1^2\right)\right] - 1\right\}, & \omega \neq 0, \\ 0.5\Delta\left(r - r_1^2 r^{-1}\right), & \omega = 0. \end{cases}$$

The inequalities $\varepsilon_1 \geqslant 0$, $\varepsilon_2 \geqslant 0$ are satisfied for $r \geqslant r_1$. There are no constraints on the disk angular velocity. The volume V_ω of the optimal disk is an increasing function of ω $V_\omega = 2\pi T\rho^{-1}\omega^{-2}\{\exp[0.5\rho\omega^2(r_2^2 - r_1^2)] - 1\}$, $\omega \neq 0$, $V_0 = \pi T(r_2^2 - r_1^2)$ and $\lim_{\omega\to 0} V_\omega = V_0$. Let us note that the solution obtained is also valid for a continuous disk when $r_1 = 0$. The condition $u = 0$ for $r = 0$ remains here from the requirement of axial symmetry. Graphs of the function V_ω/V_0 are presented in Fig. 3 for values of the parameters $r_2 = 3$, $T = \rho = \Delta = 1$. Curves 1-3 correspond to the values $r_1 = 0, 1, 2$.

In the case when forces T_k are given on both disk boundaries, we have from (5.3) by assuming that the optimal design operated in regime B

$$h = T_k r_k r^{-1} \exp\left[0.5\rho\omega^2\left(r_k^2 - r^2\right)\right], \quad k = 1, 2.$$

Therefore, the forces T_k satisfy the relationships $T_1 r_1 = T_2 r_2 \exp(r_2^2 - r_1^2)$, $T_k > 0$. From the optimality condition $u_{,r} - \rho\omega^2 ru = \Delta$ we obtain the velocity field

$$u = \Delta\left[-\int_r^{r_2} \exp\left(-0.5\rho\omega^2 r^2\right) dr - c\right] \exp\left(0.5\rho\omega^2 r^2\right),$$

$\Delta > 0$, and c are arbitrary constants. The inequalities $-\varepsilon_1 \leqslant \varepsilon_2 \leqslant 0$ then are written in the form

$$-\Delta - \rho\omega^2 ru \leqslant u/r \leqslant 0, \quad r_1 \leqslant r \leqslant r_2.$$

It hence follows that $c \geqslant 0$ and the inequality

$$\int_r^{r_2} \exp(-0{,}5\rho\omega^2 r^2)\, dr + c \leqslant \frac{r}{\rho\omega^2 r^2 + 1} \exp(-0{,}5\rho\omega^2 r^2) \tag{5.6}$$

should be satisfied for $r_1 \leqslant r \leqslant r_2$. If this inequality is satisfied for a certain r for $c \geqslant 0$, the more so for $c = 0$. Therefore we set $c = 0$. Note then a strict inequality exists for $r = r_2$ in (5.6) which also holds in a certain ring $r_1 \leqslant r \leqslant r_2$ by virtue of the continuity of the functions in (5.6). Solving the inequality (5.6) for specific values of ρ, ω, r_2, the lower bound r° can be found such that this inequality will be in any ring $r^\circ \leqslant r_1 \leqslant r \leqslant r_2$.

A detailed investigation of the inequality (5.6) is omitted for brevity. The inequality

$$\int_r^{r_2} \exp(-0{,}5\rho\omega^2 r^2)\, dr \leqslant (r_2 - r) \exp(-0{,}5\rho\omega^2 r^2)$$

evidently shows that (5.6) is a corollary of the stronger inequality

$$(r_2 - r)(\rho\omega^2 r^2 + 1) \leqslant r.$$

For a specific example we set $\rho = \omega = \Delta = 1$, $r_2 = 2$. The last inequality then takes the form

$$S(r) = -r^3 + 2r^2 - 2r + 2 \leqslant 0.$$

Since it is satisfied for $r = 2$ and $r = 1.6$ and the function $S(r)$ is monotonic in the interval [1.6;2], then it is possible to set $r_1 = 1.6$, $T_1 = \exp(1.44) \approx 4.221$, and $T_2 = 0.8$, for example. All the necessary requirements for the existence of an optimal design are satisfied here.

6. A direction associated with the construction of equally strong (equally stressed) designs has been developed actively in the last decade within the framework of searches for rational designs in addition to attempts to construct optimal designs in the sense examined above [13-17]. An analysis of their interrelationship is of interest, and we perform it here in the example of a rotating disk. If the concept of equal-strength is in agreement with the concept of equal stress in the sense of $\sigma_1 = \sigma_2 = 1$ [13], then the thickness distribution for such a design agrees with (5.5) and the difference between the corresponding solutions will be that within the framework of the formulation examined above and the optimality condition will permit determination of the velocity field also, while the formulation of the problem of an equally stressed disk will not afford such a possibility. Another approach to the construction of equally strong designs [14-16] is based on solving the elastic problem with the additional requirement of satisfying the plasticity condition in the whole domain simultaneously. If the material is incompressible and subject to the Mises plasticity condition, then an equally strong elastic design is simultaneously a minimal weight plastic design [14]. However, it is difficult to construct the corresponding solution in this case. We hence examine the problem of an equally strong design for a piecewise-linear flow condition.

We have the equilibrium equation (5.1), the relationship (5.2), and Hooke's law for an elastic disk

$$\sigma_1 = W^{-1}(\varepsilon_1 + \nu\varepsilon_2), \quad \sigma_2 = W^{-1}(\varepsilon_2 + \nu\varepsilon_1), \quad W = (1 - \nu^2)E^{-1}. \tag{6.1}$$

Here u, ε_k are the dimensionless radial displacement and principal strains, $E = E^*\sigma_0^{-1}t_0^{-1}$ (E^* is the Young's modulus), and ν is the Poisson ratio of the material. Moreover, the plasticity condition (1.3) should be satisfied for an elastic equally strong disk. From (1.3), (5.2), (6.1) we have

$$B_1 u_{,r} + B_2 u/r = W,$$

where $B_1 = a_i + \nu b_i$; $B_2 = b_i + \nu a_i$. The integral of this equation is

$$u = \begin{cases} W(B_1 + B_2)^{-1}\left(r + c_1 r^{-B_2/B_1}\right), & \text{if} \quad B_1 + B_2 \neq 0, \\ W B_1^{-1} r \ln(c_1 r), & \text{if} \quad B_1 + B_2 = 0, \end{cases} \tag{6.2}$$

where c_1 is the constant of integration. Evaluating the stresses in displacements (6.2) and substituting the latter in the equilibrium equation, we determine the disk thickness h to the accuracy of a constant c_2. The constants c_1, c_2 should be determined from the boundary conditions. In conclusion, the construction must confirm compliance with the inequalities (2.11).

In the case of an annular disk with boundary conditions (5.4), we obtain $a_i = 1$, $b_i = 0$, $B_1 = 1$, $B_2 = \nu$ by assuming that the stresses in the disk correspond to the side AB of the Treska flow condition. Taking account of the first boundary condition from (5.4), we obtain from (5.2), (6.1), and (6.2):

$$u = (1-\nu)E^{-1}\left(r - r_1^{1+\nu}r^{-\nu}\right),\quad \sigma_1 \equiv 1,\quad \sigma_2 = 1-(1-\nu)\left(r_1/r\right)^{1+\nu}.$$

Taking account of the second boundary condition from (5.4), we obtain the thickness of an equally strong plate from the equilibrium equation (5.1)

$$h = T\exp\left\{\frac{1-\nu}{1+\nu}\left[\left(\frac{r_1}{r}\right)^{1+\nu} - \left(\frac{r_1}{r_2}\right)^{1+\nu}\right] + \frac{\rho\omega^2}{2}\left(r_2^2 - r^2\right)\right\}. \qquad (6.3)$$

The inequalities (2.11) are satisfied for $r \geqslant r_1$. For $0 < r_1 \leqslant r \leqslant r_2$, $0 < \nu < 1/2$ the thickness of this design is not less than the thickness of the corresponding optimal design (5.5), where the thicknesses of both designs are equal just for $r = r_2$. For $r_1 = 0$, we have $\sigma_1 = \sigma_2 = 1$ and the thickness distributions of both designs are in agreement. Let U_ω denote the volume corresponding to the design (6.3). We present a graph of the function U_ω/V_ω as a function of r_1. Curves 1-3 in Fig. 4 correspond to $\omega = 1, 1.5, 2$ with the constants $\nu = 0.3$, $r_2 = 2$, $\rho = T = 1$. As is seen, the equally strong disk has a volume exceeding the volume of the optimal disk constructed above.

If the forces $T_k > 0$ are given on both disk boundaries, then by assuming the stresses in the disk to correspond to the side AF, we obtain $a_i = 0$, $b_i = 1$, $B_1 = \nu$, $B_2 = 1$, while $u = (1-\nu)E^{-1}(r + c_1 r^{-\alpha})$, $\alpha = 1/\nu$ from (6.2). The corresponding stresses are $\sigma_1 = 1 - c_1(\alpha - 1)r^{-1-\alpha}$, $\sigma_2 \equiv 1$. The inequalities $0 \leqslant \sigma_1 \leqslant 1$ hold for $0 \leqslant c_1 \leqslant (\alpha-1)^{-1}r_1^{\alpha+1}$. From the equilibrium equation and the boundary condition for $r = r_1$

$$h = T_1 N,\quad N = \exp\left[-\int_{r_1}^{r}\frac{\rho\omega^2 r^{3+\alpha} + c_1\alpha(\alpha-1)}{r^{2+\alpha} - c_1(\alpha-1)}\,dr\right],$$

where the constant c_1 is determined from the condition $T_2 = T_1 N|_{r=r_2}$. For example, we set $\rho = \omega = T_1 = 1$, $\nu = 1/3$, $c_1 = 1/2$, $r_1 = 2$, $r_2 = 3$. In this case the integral is easily evaluated and

$$T_2 = N|_{r=r_2} = \frac{81}{64\sqrt{2}}\exp\left(-\frac{5}{2}\right) \approx 0.074.$$

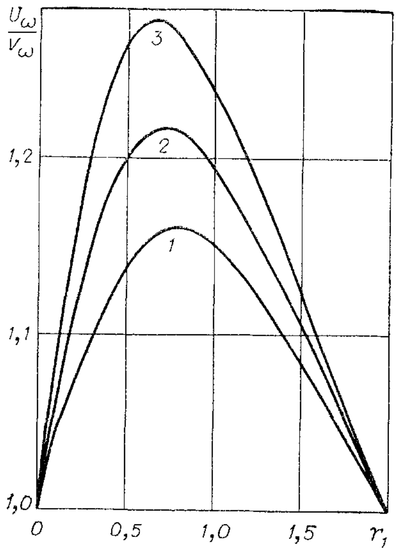

Fig. 4

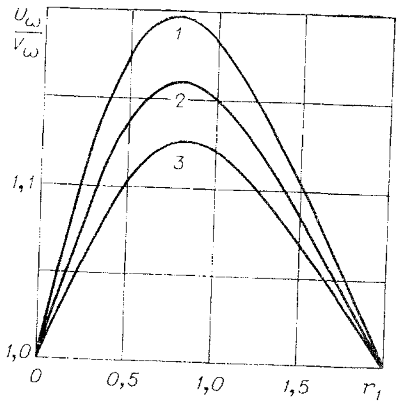

Fig. 5

The volume of the obtained equally strong elastic disk will yield the upper bound for the design of absolutely minimal weight since the stress field constructed is statically allowable [2].

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